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On the diffusive instability of some simple steady magnetohydrodynamic flows

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The stability characteristics of some simple steady magnetohydrodynamic flows within an axisymmetric container of arbitrary electrical conductivity are investigated. Attention is focused upon rapidly rotating fluids in which the unperturbed velocity and magnetic field are axially symmetric and purely zonal. Detailed solutions are obtained for the particularly simple basic state representing a rigidly rotating homogeneous fluid with a uniform axial current. The theory of dynamic (dissipationless) instabilities is reviewed and its shortcomings are elucidated. A stability criterion is derived for an inviscid fluid of small electrical conductivity within a perfectly conducting axisymmetric container and it is shown that a certain class of *inertial* modes is unstable for any non-zero magnetic field strength. When the effects of container conductivity are included it is found that a class of *slow* modes with westward phase speed may be unstable. These modes are shown to be unstable within a cylinder but appear to be stable within a sphere. The influence of density gradients within a spherical container is investigated and it is found that for a certain class of exceptional slow modes with westward phase speed, a bottom-heavy density gradient is *destabilizing*. This surprising behaviour is explained in terms of a new branch of the stability curve developed by Eltayeb & Kumar (1977).

1. Introduction

One remarkable feature of the earth's magnetic field is the persistent tendency of the non-axisymmetric portion to drift westward relative to the earth's surface. Challenged by this observation, many theoreticians have sought a reason why the asymmetric field ought to move westward rather than eastward. Current attention is centred on the hydromagnetic-wave theory, initiated by Hide (1966) and Malkus (1967) and developed further by Stewartson (1967), Acheson (1972) and others; for a review see Hide & Stewartson (1972). This theory identifies the asymmetric field and its drift with a slow magnetohydrodynamic wave of planetary scale riding on a predominantly zonal field in the earth's core. The importance of such waves had earlier been stressed by Braginskiĭ (1964), who recognized that they broke the force of Cowling's anti-dynamo theorem. Braginskiĭ did not lay particular theoretical stress on the direction of wave propagation but saw the interaction of slow waves as the primary source of the so-called α-effect, necessary to maintain the geomagnetic field by dynamo action. Braginskiĭ's ideas have been reviewed by Roberts & Soward

(1972). There are, then, two distinct objectives in the study of hydromagnetic waves in the core: to explain the westward drift and to gain some insight into the workings of the geodynamo.

In the early development of hydromagnetic-wave theory, diffusive (resistive and viscous) and buoyancy effects were ignored and attempts were made to find *dynamic* magnetic instabilities which are geophysically relevant. Hide (1966) found a class of waves which drift westward in a thin spherical shell, but subsequent work by Malkus (1967), Stewartson (1967), Wood (1977) and others failed to reveal reasons why there should be any preferred direction of propagation in a thick shell or in a full sphere for the toroidal magnetic field strengths ($\lesssim 1000$ gauss) believed to exist within the core. Malkus did find that a sufficiently strong toroidal field would be dynamically unstable to a westward-moving wave but dismissed this case as being geophysically irrelevant. Acheson (1972) showed quite generally for a cylindrical geometry that dynamically unstable waves must drift westward. He also stressed that the unreasonably large field needed to destabilize the Malkus model was a consequence of the special form of toroidal field B assumed by Malkus, namely $B \propto s$, where s is distance from the cylindrical axis of symmetry. Acheson showed that if B increases with s more rapidly than $s^{\frac{3}{2}}$, at least for cylindrical models, the system would be subject to a dynamical instability, christened a 'field-gradient instability'. We derive Acheson's criterion in § 2, and in appendix A we show how it can be generalized to an arbitrary toroidal field $B(s, z)$, arbitrary zonal velocity $v(s, z)$ and arbitrary axially symmetric density gradient.

At first sight it appears reasonable to ignore dissipation in the study of hydromagnetic waves in the core since westward drift at the current observed rate would carry the waves around the earth in less than a thousand years, whereas the free decay time of a field of planetary scale may be an order of magnitude greater. However, this argument fails to note that the waves observed today have presumably persisted over many millennia and have therefore survived dissipative losses. Thus an accurate description of the waves must include dissipative effects. With diffusion added, it is possible that dissipative effects might work selectively to the advantage of westward-propagating waves. Further, it is possible that some waves which are dynamically neutral (in the absence of dissipation) might be *diffusively* unstable when dissipative effects are introduced. As an example of this phenomenon, consider the MAC waves† studied by Braginskiĭ (1964, 1967). In the absence of diffusive effects, he found a class of waves which are convectively unstable provided the buoyancy forces are positive (i.e. the fluid is top-heavy). However, studies by Eltayeb & Roberts (1970), Eltayeb (1972, 1975), Roberts & Stewartson (1974, 1975) and Eltayeb & Kumar (1977) have clearly established that diffusive convective instabilities can occur for much smaller density gradients than can the dynamic convective instabilities found by Braginskiĭ.

We carry these ideas of diffusive destabilization further in the present paper. Primarily, we study the effect of Ohmic diffusion upon the stability of MC waves in a homogeneous fluid for a variety of container geometries and conductivities, although

† The acronym MAC stands for *M*agnetic, *A*rchimedean (buoyancy) and *C*oriolis, the three major forces which, together with pressure, control the wave dynamics. Throughout most of this paper we shall study the simpler MC waves in which buoyancy is unimportant.

the effects of viscous diffusion and density stratification are considered briefly. In the following paper Soward (1979) studies in detail the effect of Ohmic, viscous and thermal diffusion on the stability of MAC waves in a particularly simple cylindrical geometry. In what follows, we examine some simple magnetic-field configurations which are dynamically stable but which are destabilized by the introduction of small magnetic diffusivity λ (although of course the growth rate of the instability vanishes as $\lambda \to 0$). Although a general theory for these diffusive instabilities in the case of infinitely conducting container walls is presented in appendix B, attention in the main body of the paper is focused primarily on Malkus' model, $B \propto s$, in order to demonstrate most clearly the existence of these new modes of instability for a simple basic state which is predicted to be stable with respect to both Acheson's field-gradient instability and the resistive tearing instability well known in plasma physics (Furth, Killeen & Rosenbluth 1963; Gibson & Kent 1971; Baldwin & Roberts 1972). This should not, however, be construed as a belief that the field-gradient mode or the tearing mode is irrelevant to geomagnetic theory. A determination of the relative importance of these various instabilities to theory of the secular variation must await the development of a more realistic mathematical model.

This paper is organized as follows. In §2 and appendix A we present a simple 'geomagnetohydrodynamic basic state' and analyse its stability properties in the absence of dissipation, pointing out several of the shortcomings of these dynamic instabilities as they relate to the earth's core. We generalize the instability analysis in §3 and in appendix B by modelling an inhomogeneous fluid of finite conductivity within a perfectly conducting axisymmetric container and find that a new class of instabilities, which yields westwardly propagating *inertial* waves, can occur in a homogeneous inviscid fluid for any non-zero value of the magnetic field strength. The effect of viscosity upon these unstable waves is considered briefly. The stability analysis is further generalized in §4 by considering the container to have finite conductivity. Attention in this section is focused upon the particularly simple basic state consisting of a uniform axial electric current flowing through a rigidly rotating homogeneous incompressible fluid. It is found that another new class of instabilities can occur, this time yielding westwardly propagating *slow* hydromagnetic waves. These instabilities are found to occur in an inviscid fluid confined within a cylindrical container. These modes appear to be stable if the container is spherical. The effects of density gradients are considered briefly in §5. where it is found that for certain exceptional modes, a bottom-heavy density gradient may be *less* stable than a top-heavy gradient. The results are summarized in §6. In light of the complexity of the analyses of the following sections, we note here that aspects of this work have been reviewed by one of us (PHR) in three recent conference proceedings (Roberts 1977, 1978*a*, *b*).

2. Dynamic instabilities

According to one view, the magnetic field **B** of the earth is strongest in the core, where zonal fluid motions **v** wrap meridional field lines around the polar axis, creating a large toroidal field which is confined to the core. Compared with this field, the observed main geomagnetic field is dynamically ineffective and may be neglected in a first approximation. Several investigations, reviewed by Roberts & Soward (1972), have therefore concentrated on the 'geomagnetohydrodynamic basic state' in which

$$\left.\begin{aligned}\mathbf{B} = \mathbf{B}^0 = (\mu\rho_r)^{\frac{1}{2}} s\tau(s,z)\,\mathbf{1}_\phi,\\ \mathbf{v} = \mathbf{v}^0 = s\zeta(s,z)\,\mathbf{1}_\phi, \quad \rho = \rho^0 = \rho_r[1+C^0(s,z)],\end{aligned}\right\} \tag{2.1}$$

where (s,ϕ,z) are cylindrical co-ordinates with polar axis Oz, $\mathbf{1}_\phi$ is the unit vector along lines of latitude, ρ is the density and μ is the magnetic permeability. Following Braginskiĭ (1967), we have introduced the Alfvén frequency τ, the velocity shear ζ and the fractional excess in density C, where ρ_r is a reference density.

It is well known from plasma dynamics that, when $\mathbf{v} = 0$ relative to a non-rotating reference frame, state (2.1) may be subject to necking instability and to interchange instability, both of which develop on the dynamic (Alfvénic) time scale. In the geophysical context, however, the fluid in the core should closely follow the rigid-body rotation of the earth, so that $\mathbf{v} \doteqdot \boldsymbol{\Omega}\times\mathbf{r}$ in (2.1), where $\boldsymbol{\Omega} = \Omega\mathbf{1}_z$ is the angular velocity of the earth and $\mathbf{r}$ is the position vector. It has long been recognized that this motion should strongly stabilize state (2.1), particularly to axisymmetric disturbances. Therefore we shall disregard axisymmetric perturbations in this paper. Also we shall consider (2.1) to describe the basic state relative to a reference frame rotating with angular velocity $\boldsymbol{\Omega}$.

To illustrate the shortcomings of the theory of dynamic instability as it pertains to the geomagnetic field, we shall concentrate on the simple basic state in which

$$\tau = \tau(s), \quad \zeta = 0, \quad C^0 = 0. \tag{2.2}$$

The companion analysis for the more general state (2.1) may be found in appendix A. The linear stability of the basic state is examined by writing

$$\mathbf{B} = \mathbf{B}^0 + \mathbf{B}', \quad \mathbf{v} = \mathbf{v}'$$

and neglecting squares and products of all perturbation (primed) fields. Neglecting dissipation, the linearized equations are

$$\partial\mathbf{B}'/\partial t - (\mu\rho_r)^{\frac{1}{2}}\tau\,\partial_1\mathbf{v}'/\partial\phi + s(\mu\rho_r)^{\frac{1}{2}} v_s'(d\tau/ds)\,\mathbf{1}_\phi = 0, \tag{2.3}$$

$$\partial\mathbf{v}'/\partial t + 2\boldsymbol{\Omega}\times\mathbf{v}' = -\nabla p' + (\mu\rho_r)^{-\frac{1}{2}}[\tau\,\partial_1\mathbf{B}'/\partial\phi + sB_s'(d\tau/ds)\,\mathbf{1}_\phi - 2\tau\mathbf{B}'\times\mathbf{1}_z], \tag{2.4}$$

$$\nabla\cdot\mathbf{v}' = \nabla\cdot\mathbf{B}' = 0, \tag{2.5}$$

where $\partial_1/\partial\phi$ is differentiation holding $\mathbf{1}_s$ and $\mathbf{1}_\phi$ fixed.

In seeking the mode of maximum instability, we follow the method of Gilman & Cadez (unpublished) and introduce a right-handed system of local Cartesian co-ordinates (ν,ξ,ϕ) where $\mathbf{1}_\nu$ makes an arbitrary angle χ with $\mathbf{1}_z$ as shown in figure 1. We seek a condition for the system to be stable at a local point P to disturbances whose wavelength is arbitrarily short in the ξ direction. That is, we let $\partial/\partial\xi = l$ and consider $l\to\infty$. From (2.5), $v_\xi' = O(l^{-1}v_\phi')$ and, from the ξ component of (2.4),

$$p' = O(l^{-1}v_\phi').$$

Assuming the perturbations to depend on ϕ and t as $\exp[i(m\phi-\omega t)]$, the $\mathbf{1}_\phi$ and $\mathbf{1}_\nu$ components of (2.3) and (2.4) are seen to possess non-trivial solutions provided that

$$(m^2\tau^2-\omega^2)\,[m^2\tau^2-\omega^2-s(d\tau^2/ds)\sin^2\chi] = 4[\Omega\omega+m\tau^2]^2\sin^2\chi. \tag{2.6}$$

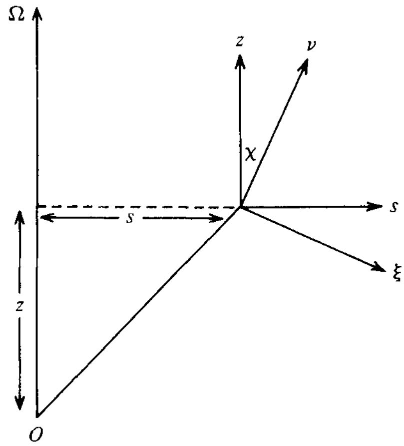

FIGURE 1. Relationship between the local Cartesian co-ordinates (ν, ξ, ϕ) and the polar co-ordinates (s, ϕ, z).

We may regard this as a local test of stability. If we can find a direction $\mathbf{1}_\nu$ at any point P such that (2.6) yields a mode with $\operatorname{Im}\omega > 0$, we may infer that the state (2.2) is unstable. If no such direction can be found, no matter where P is chosen, the state is stable to localized disturbances, although, of course, there is no guarantee that it will be stable to modes of finite wavelength. In this sense, (2.6) provides only a necessary condition for stability. In two cases studied in §4 the condition is sufficient as well as necessary. We should not wish to claim this is true for the more general case studied in appendix A, in which $\zeta = 0$. It is well known that local tests are unreliable guides to the stability of purely hydrodynamic shear flows, tending to predict stability when large-scale instability exists. This suggests that the general criterion of appendix A provides only a sufficient condition for linear instability.

It is well known that for large Ω the normal modes of linear perturbations fall naturally into two classes: inertial modes with small superimposed magnetic and buoyancy effects and MAC waves. For the MAC modes, we may neglect the inertial terms, simplifying (2.6) to

$$m^2\tau^2[m^2\tau^2 - s(d\tau^2/ds)\sin^2\chi] = 4(\Omega\omega + m\tau^2)^2\sin^2\chi. \tag{2.7}$$

A sufficient condition for instability is evidently

$$m^2\tau^2 < s(d\tau^2/ds)\sin^2\chi \tag{2.8}$$

for some direction $\mathbf{1}_\nu$ at some point P. It is apparent that the most unstable mode has a zonal wavenumber $m = 1$ and the most unstable direction is $\mathbf{1}_\nu = \mathbf{1}_s$, giving

$$0 < d(\tau^2/s)/ds. \tag{2.9}$$

In the absence of shear, instability occurs only at points where $s^{-\frac{1}{2}}B^0$ increases with increasing radius. Acheson calls these 'field-gradient instabilities'. A more general criterion for dynamic stability involving shear and buoyancy effects is developed in appendix A. In later sections, we shall pay particular attention to the basic state in which

$$\tau = \text{constant}, \quad \zeta = C^0 = 0. \tag{2.10}$$

According to (2.9), this state is dynamically stable with respect to MAC modes or, more precisely, MC modes. Also, according to (A 17), inertial instabilities of (2.10) will occur if τ anywhere exceeds Ω, but such a value of τ is too great to be of geophysical interest. We shall see, however, in §4 that state (2.10) is prone to secular instabilities and may yield an MC wave with a westward phase speed.

The foregoing analysis clearly illustrates a fundamental shortcoming of dissipationless models first noted by Roberts & Stewartson (1974): the most unstable modes possess infinitesimal wavelengths, which is obviously incompatible with the neglect of dissipation. Intuitively one might anticipate that the addition of dissipation to the model would set things right by shifting the most unstable mode to a finite wavelength. However, a moment's reflexion reveals further shortcomings of the dissipationless models which cannot be rectified by the addition of dissipation as we shall now explain. Because of the neglect of diffusive effects, the instabilities do not give rise to persistent circulatory motions but rather result in a single convulsive overturning to a state of lower energy. Further, this instability acts on a rapid Alfvénic time scale. The addition of a small amount of dissipation cannot greatly alter this type of instability. If a single dynamic overturning is not the desired type of instability and addition of dissipation does not change the type, what then is the correct picture?

As we shall see, this dilemma is resolved by the occurrence of a new class of instabilities which grow on a much longer (diffusive) time scale and thus may result in persistent motions. What is more, these instabilities can occur with much weaker magnetic fields than can the dynamic instabilities just described, making it doubtful whether dynamic instabilities play any role at all in the hydromagnetics of rotating fluids (provided $\Omega d^2 \gg \lambda$, where d is a typical length scale; this is always the case in the systems we are discussing). In the following sections we shall add Ohmic dissipation to our model and demonstrate the existence of these diffusive instabilities.

3. Diffusive instability within an axisymmetric perfectly conducting container

In this section we consider the stability of a fluid of finite electrical conductivity confined within an axisymmetric container which is a perfect conductor. We shall find a set of neutrally stable dissipationless modes with frequency ω_0, then determine the perturbation ω_1 to this frequency as a small amount of magnetic diffusivity λ is introduced within the fluid. We shall find that certain modes ω_0 which are predicted to be stable by the theory of §2 have $\operatorname{Im}\omega_1 > 0$, indicating diffusive instability. Our aim is not to locate the most unstable mode but merely to demonstrate that a class of diffusively unstable modes can occur within an axisymmetric perfectly conducting container. In this section we shall limit our attention to the particularly simple basic state (2.10), for which explicit solution of ω_1 is possible. As in §2, a companion analysis for the more general state (2.1) may be found in appendix B.

One advantage of state (2.10) is that it does not evolve on the diffusive time scale after Ohmic diffusion is introduced into the problem since $\nabla^2\mathbf{B}^0 = 0$. However, the electric field associated with $\mathbf{B}^0$ in the fluid is not in general compatible with the infinite conductivity of the boundary; we must artificially specify a suitable distribution of potential differences over the surface W. To exclude the possibility that these sources feed energy into the perturbations directly, we require that they be independent of the perturbations $\mathbf{B}'$, $\mathbf{v}'$, etc. With the container specified to be a perfect conductor, we simply require $\mathbf{n}\times\mathbf{E}' = 0$, where $\mathbf{n}$ is the unit normal and $\mathbf{E}'$ is the electric field perturbation. Evidently the Poynting flux of energy into the fluid is the same whether the basic state is perturbed or not. Since $\mathbf{n}\,.\,\mathbf{v}' = 0$ on W, the vanishing of $\mathbf{n}\times\mathbf{E}'$ implies that $\mathbf{n}\times(\nabla\times\mathbf{B}') = 0$ on W. The boundary conditions (which are not all independent; see Roberts 1967, p. 24) are thus

$$\mathbf{n}.\mathbf{B}' = 0, \quad \mathbf{n}\times(\nabla\times\mathbf{B}') = 0, \quad \mathbf{n}.\mathbf{v}' = 0 \quad \text{on} \quad W. \tag{3.1}$$

The perturbations satisfy (2.3)–(2.5) with the exception that the dissipation term $\lambda\nabla^2\mathbf{B}'$ must be added to the right-hand side of (2.3). These equations may be made dimensionless by the following scaling: length d, time Ω^{-1}, velocity Ωd, pressure $\rho_r\Omega^2 d^2$ and magnetic field $(\mu\rho_r)^{\frac{1}{2}}\,\Omega d$. Assuming harmonic dependence on ϕ and t of the form $\exp[i(m\phi-\omega t)]$, we obtain

$$-i\omega\mathbf{v}' + 2\mathbf{1}_z\times\mathbf{v}' = -\nabla p' + \tau^2[im\mathbf{B}' + 2\mathbf{1}_z\times\mathbf{B}'], \tag{3.2}$$

$$-i\omega\mathbf{B}' - im\mathbf{v}' = \Lambda\nabla^2\mathbf{B}', \tag{3.3}$$

$$\nabla.\mathbf{v}' = \nabla.\mathbf{B}' = 0, \tag{3.4}$$

where

$$\Lambda \equiv \lambda/\Omega d^2. \tag{3.5}$$

Note that with the simple state (2.10) all effects involving C^0, ζ or $\nabla\tau$ are absent; the more general state (2.1) is analysed in appendix B. Also note that ω and τ here are equal to ω/Ω and τ/Ω of §2.

The induction equation (3.3) may be satisfied by writing

$$\mathbf{v}' = (-\omega\mathbf{B}' + i\Lambda\nabla^2\mathbf{B}')/m. \tag{3.6}$$

Now the problem is simplified to

$$[m^2\tau^2 - \omega^2 + i\Lambda\nabla^2]\,\mathbf{B}' + 2i[-\omega - m\tau^2 + i\Lambda\nabla^2]\,\mathbf{1}_z\times\mathbf{B}' = -im\nabla p', \tag{3.7}$$

$$\nabla.\mathbf{B}' = 0 \tag{3.8}$$

with boundary conditions

$$\mathbf{n}.\mathbf{B}' = 0, \quad \mathbf{n}\times(\nabla\times\mathbf{B}') = 0 \quad \text{on} \quad W. \tag{3.9}$$

Let us divide the perturbations into a neutral mode designated by the subscript 0 and a dissipative perturbation designated by the subscript 1:

$$\mathbf{B}' = \mathbf{B}_0 + \mathbf{B}_1, \quad p' = p_0 + p_1, \quad \omega = \omega_0 + \omega_1 \tag{3.10}$$

where $\mathbf{B}_0$, p_0 and ω_0 are solutions of the problem

$$\mathbf{B}_0 - iR\mathbf{1}_z\times\mathbf{B}_0 = -im\nabla Q_0, \tag{3.11}$$

$$\nabla.\mathbf{B}_0 = 0, \tag{3.12}$$

with

$$\mathbf{n}.\mathbf{B}_0 = 0 \quad \text{on} \quad W, \tag{3.13}$$

where

$$R \equiv 2(\omega_0 + m\tau^2)/(m^2\tau^2 - \omega_0^2), \quad Q_0 \equiv p_0/(m^2\tau^2 - \omega_0^2). \tag{3.14}$$

The parameter R, which is equivalent to the factor $2/\lambda$ employed by Malkus [1967, equation (2.27)] effectively replaces ω_0 as the eigenvalue; solving for ω_0 in terms of R yields

$$\omega_0 = [-1 \pm (1 + a\tau^2)^{\frac{1}{2}}]/R, \tag{3.15}$$

where

$$a \equiv mR(mR-2). \tag{3.16}$$

With τ constant, we see that (3.11) and (2.6) are equivalent provided that

$$R^2 \sin^2 \chi = 1. \tag{3.17}$$

From this it is apparent that $|R| \geqslant 1$. The solution of (3.15) with the top sign will be referred to as the slow or MC mode while that with the bottom sign will be referred to as the fast or inertial mode, although this distinction is clearcut only if $\tau^2 \ll 1$. The factor under the square root in (3.15) is less than unity if $a < 0$ and, by (3.16), a is negative whenever $m = 1$ and $1 < R < 2$. Eigenmodes for which $a \geqslant 0$ will be referred to as ordinary modes and those for which $a < 0$ will be referred to as *exceptional* modes because, as we shall see, the latter modes have exceptional stability properties. It may be seen that the simple basic state (2.10) is dynamically unstable if

$$a < -1/\tau^2. \tag{3.18}$$

The smallest value of τ^2 for dynamic instability occurs if $m = R = 1$, giving $\tau^2 = 1$. In the geophysical context, $\tau^2 \ll 1$ and (3.18) shows that the basic state is stable with respect to the dynamic mode of instability.

The dissipative perturbation satisfies

$$\begin{aligned}(m^2\tau^2-\omega_0^2)\,[\mathbf{B}_1 - iR\mathbf{1}_z \times \mathbf{B}_1] + im\nabla p_1 \\ + i\Lambda\nabla^2[\omega_0(\mathbf{B}_0+\mathbf{B}_1) + 2i\mathbf{1}_z \times (\mathbf{B}_0+\mathbf{B}_1)] \\ = 2\omega_1[\omega_0 \mathbf{B}_0 + i\mathbf{1}_z \times \mathbf{B}_0],\end{aligned} \tag{3.19}$$

$$\nabla . \mathbf{B}_1 = 0 \tag{3.20}$$

with

$$\mathbf{n}.\mathbf{B}_1 = 0, \quad \mathbf{n} \times [\nabla \times (\mathbf{B}_0+\mathbf{B}_1)] = 0 \quad \text{on} \quad W. \tag{3.21}$$

Because (3.11)–(3.13) admit a homogeneous solution, (3.19)–(3.21) will not be solvable unless a certain consistency condition is satisfied; this condition will serve to determine ω_1 without actually solving for $\mathbf{B}_1$ and p_1. The consistency condition is obtained by scalar multiplying (3.19) by $\mathbf{B}_0^*$, where an asterisk as a superscript denotes a complex conjugate, and integrating the result over the volume V of the fluid. Making use of the conjugate of (3.11) and the facts that

$$\int_V (\mathbf{B}_0^* . \nabla p_1 + \mathbf{B}_1 . \nabla p_0^*)\, dV = \int_W (p_1 \mathbf{B}_0^* + p_0^* \mathbf{B}_1) . \mathbf{n}\, dW = 0$$

and

$$\nabla^2[\mathbf{1}_z \times (\mathbf{B}_0+\mathbf{B}_1)] = \mathbf{1}_z \times [\nabla^2(\mathbf{B}_0+\mathbf{B}_1)],$$

the consistency integral may be expressed as

$$\begin{aligned}\int_W \mathbf{n} . [i\omega_0 \mathbf{B}_0^* \times \nabla \times (\mathbf{B}_0+\mathbf{B}_1) + 2(\mathbf{B}_0^* \times \mathbf{1}_z) \times \nabla \times (\mathbf{B}_0+\mathbf{B}_1)]\, dW \\ + \int_V [(i\omega_0 \nabla \times \mathbf{B}_0^* + 2\nabla \times (\mathbf{B}_0^* \times \mathbf{1}_z)] . \nabla \times (\mathbf{B}_0+\mathbf{B}_1)\, dV \\ = -2(\omega_1/\Lambda) \int_V [\omega_0 |\mathbf{B}_0|^2 + i\mathbf{1}_z \times \mathbf{B}_0 . \mathbf{B}_0^*]\, dV.\end{aligned} \tag{3.22}$$

With the bounding surface W being a perfect conductor, $\nabla \times (\mathbf{B}_0 + \mathbf{B}_1)$ is parallel to $\mathbf{n}$, making the surface integrals in (3.22) identically zero. Outside any boundary layers $|\nabla \times \mathbf{B}_1| \ll |\nabla \times \mathbf{B}_0|$ and may be neglected in the volume integrals in (3.22). With a perfectly conducting boundary, the boundary layer in the fluid near W is weaker than for finitely conducting boundaries; specifically, $|\nabla \times \mathbf{B}_1| = O(|\nabla \times \mathbf{B}_0|)$. However, we need only the volume integral of $\nabla \times \mathbf{B}_1$ in (3.22) and the contribution from a thin boundary layer is small and may be neglected. Further, it follows from (3.11) that

$$iR \int_V \mathbf{1}_z \times \mathbf{B}_0 . \mathbf{B}_0^* \, dV = \int_V |\mathbf{B}_0|^2 \, dV \tag{3.23}$$

and

$$-iR \int_V \nabla \times (\mathbf{B}_0^* \times \mathbf{1}_z) . \nabla \times \mathbf{B}_0 \, dV = \int_V |\nabla \times \mathbf{B}_0|^2 \, dV = k \int_V |\mathbf{B}_0|^2 \, dV,$$

say, where $k > 0$. Now (3.22) is simply

$$2(1 + R\omega_0)\,\omega_1 = -ik\Lambda(2 + R\omega_0). \tag{3.24}$$

Instability occurs if $\operatorname{Im} \omega_1 > 0$ or if

$$(2 + R\omega_0)/(1 + R\omega_0) < 0. \tag{3.25}$$

Instability is clearly impossible if $R\omega_0 > 0$. From (3.15) we see that (3.25) can be satisfied only if we choose the lower sign, making $1 + R\omega_0$ negative and hopefully $2 + R\omega_0$ positive. In order that $2 + R\omega_0$ be positive we need the factor in the square root of (3.15) to be less than unity or equivalently

$$a < 0. \tag{3.26}$$

This inequality is satisfied if

$$m = 1, \quad 1 < R < 2. \tag{3.27}$$

Thus we have found that the particularly simply basic state (2.10), which represents a rigidly rotating homogeneous incompressible fluid with a uniform axial electric current, is *unstable to a westwardly propagating inertial wave*. This instability occurs for *any non-zero value of the electric current*, provided that the magnetic diffusivity of the fluid is small (more accurately, provided that $\mu\rho\lambda\Omega/B^2$ is small), the magnetic diffusivity of the axisymmetric container is much smaller than that of the fluid and the fluid is inviscid. It should be remarked that this conclusion was reached without explicit solution of the neutral-eigenmode problem (3.11)–(3.13). The instability is most pronounced for $R = 1+$, which gives $\omega_0 = 2$.

The discovery of a class of unstable inertial modes is rather unexpected. Intuition tells one that the addition of dissipation to a neutral wave causes it to decay secularly, and (3.25) predicts just that behaviour provided that $(2 + R\omega_0)/(1 + R\omega_0)$ is positive, which it is for almost all eigenvalues m and R. It is only in the range $m = 1$, $1 < R < 2$ that the sense of the secondary currents, which normally help dissipate the neutral mode, becomes reversed and leads to enhancement. It is of interest to note that if $m = 1$ and $1 < R < 2$, $\omega_0 < 0$ for either sign in (3.15), indicating that only westward waves can exhibit this form of instability. We shall see in the following section that this is not an isolated phenomenon; we shall find yet another set of unstable modes which yield westwardly propagating waves.

4. Diffusive instability within an axisymmetric container of arbitrary conductivity

In the previous section we found a class of exceptional waves which, while stable with respect to dynamic modes of instability, exhibit an unexpected mode of diffusive instability which yields westwardly propagating inertial waves. That analysis was performed assuming the axisymmetric container to be a perfect conductor. In this section we investigate the effect of finite resistivity of the container and find *another* new mode of diffusive instability which yields a westwardly propagating *slow* wave.

If the container has finite conductivity, the discussion following (3.22) is invalid and accurate evaluation of the integrals in (3.22) requires detailed analysis of the boundary-layer structure near W and solution of the eigenvalue problem (3.11)–(3.13). In this section we shall proceed with this task for the case of an axisymmetric container of arbitrary conductivity.

The simple basic state (2.10) satisfies $\nabla^2\mathbf{B}^0 = 0$ and hence does not require any volume sources of magnetic field for its maintenance. In general, the state still requires a suitable distribution of potential differences over the surface W. However, there are two special cases for which the basic state can be maintained by an externally applied uniform potential difference: a cylindrical container of arbitrary conductivity and an axisymmetric container with conductivity equal to that of the fluid. These two cases are important because they are, in theory at least, experimentally realizable. However the parameter ranges needed make experimental verification a formidable task, on a par with modelling a hydromagnetic dynamo in the laboratory. We shall ascertain the stability of the basic state (2.10) for two container geometries of particular interest, a cylinder and a sphere, and show that the exceptional modes, $m = 1$ and $1 < R < 2$, are unstable within a cylinder but appear to be stable within a sphere.

With the addition of boundary resistivity, the governing equations within the fluid remain (3.2)–(3.4) but the boundary conditions become

$$\mathbf{n}.\mathbf{v} = 0, \quad \mathbf{B} = \mathbf{B}_c, \quad \mathbf{n}\times(\nabla\times\mathbf{B}) = \Gamma\mathbf{n}\times(\nabla\times\mathbf{B}_c) \quad \text{on} \quad W, \tag{4.1}$$

where

$$\Gamma \equiv \lambda_c/\lambda \tag{4.2}$$

and a subscript c denotes parameters and variables associated with the container. We now need to determine the magnetic field within the container by solving

$$-i\omega\mathbf{B}_c = \Gamma\Lambda\nabla^2\mathbf{B}_c \tag{4.3}$$

subject to the further condition that the magnitude of $\mathbf{B}_c$ remains bounded as the distance from W increases. Eliminating $\mathbf{v}'$ by (3.6), the equations in the volume V again are (3.7) and (3.8), while the boundary conditions now are

$$\omega\mathbf{n}.\mathbf{B} = i\Lambda\mathbf{n}.\nabla^2\mathbf{B}, \quad \mathbf{B} = \mathbf{B}_c, \quad \mathbf{n}\times(\nabla\times\mathbf{B}) = \Gamma\mathbf{n}\times(\nabla\times\mathbf{B}_c). \tag{4.4}$$

The first step is to solve the exterior equation (4.3) and use the result to express conditions (4.4) entirely in terms of $\mathbf{B}$. Let us introduce a normal co-ordinate function $\beta(s,z)$ with the property that

$$\beta = \beta_0 \quad \text{on} \quad W \tag{4.5}$$

and $\beta > \beta_0$ outside V, where β_0 is a constant. By (4.5), $\mathbf{n}$ is parallel to $\nabla\beta$. To simplify the solution of (4.3) we shall assume that $|\omega| \gg \Gamma\Lambda$ so that (4.3) simplifies to the boundary-layer equations

$$\left.\begin{aligned} -i\omega h_\beta^2 \mathbf{B}_c &= \Gamma\Lambda\, \partial^2\mathbf{B}_c/\partial\beta^2, \\ s\,\partial B_{c\beta}/\partial\beta + h^{-1}\partial(sh_\beta B_{c\gamma})/\partial\gamma + h_\beta imB_{c\phi} &= 0, \end{aligned}\right\} \tag{4.6}$$

where γ is a third co-ordinate such that (β, γ, ϕ) form a right-handed orthogonal system on W. The parameters h_β and h_γ are the scale factors for the co-ordinates β and γ. For cylindrical co-ordinates $\beta = s$, $h_\beta = 1$, $\beta_0 = 1$, $\gamma = -z$ and $h_\gamma = 1$ while for spherical co-ordinates $\beta = r$, $h_\beta = 1$, $\beta_0 = 1$, $\gamma = \theta$ and $h_\gamma = r$. The solution of (4.6) may be expressed for $\operatorname{Re} q \gg 1$ as

$$\left.\begin{aligned} B_{c\phi} &= B_\phi \exp[-q_c(\beta-\beta_0)], \quad B_{c\gamma} = B_\gamma \exp[-q_c(\beta-\beta_0)], \\ B_{c\beta} &= (sq_c)^{-1}[imh_\beta B_\phi + h_\gamma^{-1}\partial(sh_\beta B_\gamma)/\partial\gamma]\exp[-q_c(\beta-\beta_0)], \end{aligned}\right\} \tag{4.7}$$

where

$$q_c = h_\beta(-i\omega/\Gamma\Lambda)^{\frac{1}{2}} = (1-\sigma_\omega i)\,|\omega/2\Gamma\Lambda|^{\frac{1}{2}} h_\beta, \quad \sigma_\omega = \operatorname{sgn}\omega, \tag{4.8}$$

and we have used $(4.4)_3$ in part. Substitution of (4.7) into (4.4) yields

$$\left.\begin{aligned} \omega \mathbf{1}_\beta.\mathbf{B} &= i\Lambda \mathbf{1}_\beta.\nabla^2\mathbf{B}, \\ \partial(sB_\phi)/\partial\beta = -\Gamma q_c sB_\phi, &\quad \partial(h_\gamma B_\gamma)/\partial\beta = -\Gamma q_c h_\gamma B_\gamma \end{aligned}\right\} \text{ on } W. \tag{4.9}$$

In writing (4.9) we have used the fact that $|B_\beta| \ll |B_\phi|$ or $|B_\gamma|$. Also we have neglected the fourth condition, which is redundant. If $\Gamma \to 0$, (4.9) reduce to (3.9) as they should.†

As in § 3, we may divide the perturbation into neutral and dissipative parts by (3.10), with the neutral mode satisfying (3.11)–(3.13) and the dissipative mode satisfying (3.19) and (3.20) subject to

$$\left.\begin{aligned} \omega_0 \mathbf{1}_\beta.\mathbf{B}_1 &= i\Lambda\mathbf{1}_\beta.\nabla^2(\mathbf{B}_0+\mathbf{B}_1) \\ \partial(sB_{0\phi}+sB_{1\phi})/\partial\beta &= -\Gamma q_c(sB_{0\phi}+sB_{1\phi}) \\ \partial(h_\gamma B_{0\gamma}+h_\gamma B_{1\gamma})/\partial\beta &= -\Gamma q_c(h_\gamma B_{0\gamma}+h_\gamma B_{1\gamma}) \end{aligned}\right\} \text{ on } W. \tag{4.10}$$

As in § 3, we may determine ω_1 from a consistency integral formed by scalar multiplying (3.19) by $\mathbf{B}_0^*$ and integrating over V. After some manipulation paralleling that of § 3 the integral may be expressed as

$$2(1+R\omega_0)\,\omega_1 = -i\Lambda k(2+R\omega_0) + \Lambda^{\frac{1}{2}}R\omega_0 I/\tau, \tag{4.11}$$

where

$$\begin{aligned} I = \Lambda^{\frac{1}{2}}\tau\left[\int_V |\mathbf{B}_0|^2\,dV\right]^{-1} \int_W \mathbf{1}_\beta.\{&-(mp_0^*/\omega_0^2)\,\nabla^2(\mathbf{B}_0+\mathbf{B}_1) \\ &+ i\mathbf{B}_0^* \times \nabla \times [\mathbf{B}_0+\mathbf{B}_1+(2i/\omega_0)\,\mathbf{1}_z \times (\mathbf{B}_0+\mathbf{B}_1)]\}\,dW. \end{aligned} \tag{4.12}$$

Equation (4.11) is a generalization of (3.24) and reduces to it in the limit $\Gamma \to 0$.

Following the argument presented in § 3, only the boundary-layer portion of $\mathbf{B}_1$ is important in the consistency integral (4.12). Assuming that

$$\mathbf{B}_1 = \mathbf{B}_1^I + \tilde{\mathbf{B}}_1, \tag{4.13}$$

† This may be seen by noting that $\mathbf{1}_\beta.\nabla^2\mathbf{B} = \nabla.[\mathbf{1}_\beta \times (\nabla\times\mathbf{B})]$ and $\mathbf{1}_\beta \times (\nabla \times \mathbf{B}) = 0$ on W if $\Gamma = 0$.

where the superscript I denotes the interior contribution and the tilde denotes the boundary-layer contribution, and that

$$\tilde{\mathbf{B}}_1 = \mathbf{b}\exp[q(\beta-\beta_0)], \tag{4.14}$$

(3.19) gives, to dominant order,

$$\left.\begin{aligned}
&[m^2\tau^2-\omega_0^2+i\Lambda\omega_0 q^2/h_\beta^2]\,b_\gamma+[iR(m^2\tau^2-\omega_0^2)+2\Lambda q^2/h_\beta^2]\,\psi b_\phi = 0,\\
&[m^2\tau^2-\omega_0^2+i\Lambda\omega_0 q^2/h_\beta^2]\,b_\phi-[iR(m^2\tau^2-\omega_0^2)+2\Lambda q^2/h_\beta^2]\,\psi b_\gamma = 0,\\
&sqb_\beta+h_\gamma^{-1}\partial(sh_\beta b_\gamma)/\partial\gamma+imh_\beta b_\phi = 0,
\end{aligned}\right\} \tag{4.15}$$

where

$$\psi \equiv \mathbf{1}_z.\mathbf{1}_\beta. \tag{4.16}$$

If we have a cylinder, $\psi = 0$, while $\psi = \cos\theta$ for a sphere. The characteristic equation is

$$[m^2\tau^2-\omega_0^2+i\Lambda\omega_0 q^2/h_\beta^2]^2+[iR(m^2\tau^2-\omega_0^2)+2\Lambda q^2/h_\beta^2]^2\psi^2 = 0. \tag{4.17}$$

Taking the square root twice, we obtain two modes with positive real part:

$$q_\pm = (1+\sigma_\pm i)\,h_\beta|1\pm R\psi|^{\frac{1}{2}}\,|m^2\tau^2-\omega_0^2|^{\frac{1}{2}}\,|\omega_0\mp 2\psi|^{-\frac{1}{2}}\,(2\Lambda)^{-\frac{1}{2}}, \tag{4.18}$$

where

$$\sigma_\pm = \operatorname{sgn}[(1\pm R\psi)\,(\omega_0\mp 2\psi)\,(m^2\tau^2-\omega_0^2)]. \tag{4.19}$$

In light of (4.18), we may replace (4.14) by

$$\left.\begin{aligned}
\tilde{B}_{1\gamma} &= b_+\exp[q_+(\beta-\beta_0)]+b_-\exp[q_-(\beta-\beta_0)],\\
\tilde{B}_{1\phi} &= -ib_+\exp[q_+(\beta-\beta_0)]+ib_-\exp[q_-(\beta-\beta_0)]
\end{aligned}\right\} \tag{4.20}$$

and obtain the third component from $(4.15)_3$:

$$\begin{aligned}
\tilde{B}_{1\beta} = &-(sq_+)^{-1}[h_\gamma^{-1}\partial(sh_\beta b_+)/\partial\gamma+mh_\beta b_+]\exp[q_+(\beta-\beta_0)]\\
&-(sq_-)^{-1}[h_\gamma^{-1}\partial(sh_\beta b_-)/\partial\gamma-mh_\beta b_-]\exp[q_-(\beta-\beta_0)].
\end{aligned} \tag{4.21}$$

The functions $b_\pm$ may be determined by satisfaction of conditions $(4.10)_2$ and $(4.10)_3$. Noting that the derivatives of $\mathbf{B}_0$ are negligibly small, we obtain

$$b_\pm = -\tfrac{1}{2}A_\pm\Gamma q_c/(\Gamma q_c+q_\pm), \tag{4.22}$$

where

$$A_\pm = B_{0\gamma}\pm iB_{0\phi}. \tag{4.23}$$

Now on W,

$$\mathbf{1}_\beta.\nabla^2(\mathbf{B}_0+\mathbf{B}_1) \doteq -\frac{q_+}{sh_\beta^2}\left[\frac{1}{h_\gamma}\frac{\partial(sh_\beta b_+)}{\partial\gamma}+mh_\beta b_+\right]-\frac{q_-}{sh_\beta^2}\left[\frac{1}{h_\gamma}\frac{\partial(sh_\beta b_+)}{\partial\gamma}-mh_\beta b_-\right],$$

$$\mathbf{1}_\beta.\mathbf{B}_0^*\times\nabla\times\tilde{\mathbf{B}}_1 = [b_+q_+(B_{0\gamma}^*-iB_{0\phi}^*)+b_-q_-(B_{0\gamma}^*+iB_{0\phi}^*)]/h_\beta,$$

$$\mathbf{1}_\beta.\mathbf{B}_0^*\times\nabla\times(\mathbf{1}_z\times\tilde{\mathbf{B}}_1) = \psi[b_+q_+(B_{0\phi}^*+iB_{0\gamma}^*)+b_-q_-(B_{0\phi}^*-iB_{0\gamma}^*)]/h_\beta,$$

and (4.12) may be expressed as

$$\begin{aligned}
I = \frac{\Lambda^{\frac{1}{2}}\tau}{\int_V|\mathbf{B}_0|^2dV}\int_W\Bigg\{&\frac{m}{s\omega_0^2h_\beta^2}\left[\frac{q_+}{h_\gamma}\frac{\partial(sh_\beta b_+)}{\partial\gamma}+\frac{q_-}{h_\gamma}\frac{\partial(sh_\beta b_-)}{\partial\gamma}\right.\\
&\left.-\tfrac{1}{2}mh_\beta q_c(\delta_+A_+-\delta_-A_-)\right]p_0^*+i\frac{q_c}{h_\beta}\left(\frac{\psi}{\omega_0}-\tfrac{1}{2}\right)\delta_+A_+A_+^*\\
&-i\frac{q_c}{h_\beta}\left(\frac{\psi}{\omega_0}+\tfrac{1}{2}\right)\delta_-A_-A_-^*\Bigg\}\,dW,
\end{aligned} \tag{4.24}$$

where

$$\delta_\pm = \Gamma q_\pm/(\Gamma q_c + q_\pm). \tag{4.25}$$

Note that

$$A_\pm A_\pm^* = |\mathbf{B}_0|^2 \pm i\mathbf{1}_\beta . \mathbf{B}_0^* \times \mathbf{B}_0.$$

In writing (4.24) we have retained only the boundary-layer contributions $\tilde{\mathbf{B}}_1$ in the surface integral. If the container is a perfect conductor, $b_\pm$ and $\delta_\pm = 0$ and we recover (3.24). Consistency condition (4.11) represents a test of the stability of the neutral mode $(\mathbf{B}_0, p_0, \omega_0)$ when dissipation is introduced. To make further progress, we must solve (3.11)–(3.13) for the neutral mode. We shall do this for two container geometries of particular interest: a cylinder and a sphere.

4.1. *A cylindrical container*

If the container is a cylinder, $\beta = s$, $\gamma = -z$, $\beta_0 = 1$, $h_\gamma = 1$, $\psi = 0$, $h_\beta = 1$ and $\mathbf{1}_\beta = \mathbf{1}_s$. Also,

$$\left.\begin{aligned} q_\pm = q = (1+i\sigma)\,|m^2\tau^2 - \omega_0^2|^{\frac{1}{2}}\,|2\Lambda\omega_0|^{-\frac{1}{2}}, \\ \sigma = \operatorname{sgn}[\omega_0(m\tau^2 - \omega_0^2)], \quad \delta_\pm = \delta = \Gamma q/(\Gamma q_c + q). \end{aligned}\right\} \tag{4.26}$$

With a cylindrical container, we may assume harmonic dependence in the axial direction of the form $\exp(inz)$. In this case, (4.11) is simply

$$[(2(1+R\omega_0)\,\omega_1 + i\Lambda k(2+R\omega_0)]\int_V |\mathbf{B}_0|^2 dV$$
$$= -iR\Lambda q_c \delta \int_W [(mB_{0\phi} + nB_{0z})\,mp_0^*/\omega_0 + \omega_0|\mathbf{B}_0|^2]\,dW. \tag{4.27}$$

Noting that

$$\nabla \times (\mathbf{1}_z \times \mathbf{v}) = -\partial \mathbf{v}/\partial z,$$

where $\mathbf{v}$ is a solenoidal vector, the curl of (3.11) may be expressed as

$$\nabla \times \mathbf{B}_0 + iR\,\partial\mathbf{B}_0/\partial z = 0. \tag{4.28}$$

Further, the curl of (4.28) is

$$\nabla^2\mathbf{B}_0 = R^2\,\partial^2\mathbf{B}_0/\partial z^2. \tag{4.29}$$

The axial component of this equation is

$$d^2B_{0z}/dz^2 + dB_{0z}/s\,ds + [n^2(R^2-1) - m^2/s^2]\,B_{0z} = 0. \tag{4.30}$$

This is a Bessel equation; the solution which is finite as $s = 0$ is

$$B_{0z} = J_m(\xi s)/J_m(\xi), \tag{4.31}$$

where

$$\xi \equiv n(R^2-1)^{\frac{1}{2}}, \tag{4.32}$$

and we have set the amplitude of B_{0z} at $s = 1$ equal to unity without loss of generality. Using (4.28), we have

$$B_{0s} = in[mRB_{0z} + s\,dB_{0z}/ds]/\xi^2 s. \tag{4.33}$$

Using standard formulae plus

$$J_{m\pm1}(\xi) = m(1 \pm R)\,J_m(\xi)/\xi,$$

we have

$$\left.\begin{aligned} B_{0s} &= \frac{im}{2n}\left[\frac{J_{m+1}(\xi s)}{J_{m+1}(\xi)} - \frac{J_{m-1}(\xi s)}{J_{m-1}(\xi)}\right], \\ B_{0\phi} &= \frac{m}{2n}\left[\frac{J_{m+1}(\xi s)}{J_{m+1}(\xi)} + \frac{J_{m-1}(\xi s)}{J_{m-1}(\xi)}\right], \\ p_0 &= \frac{m^2\tau^2-\omega_0^2}{mn}\frac{J_m(\xi s)}{J_m(\xi)}. \end{aligned}\right\} \tag{4.34}$$

Substitution of (4.31) and (4.34) into (4.27) and integration yield

$$[2(1+R\omega_0)\,\omega_1 + i\Lambda n^2R^2(2+R\omega_0)]\,[R(m^2+n^2)-m] = -i\Lambda q_c\delta(R^2-1)\,(m^2+n^2)\,m^2\tau^2/\omega_0. \tag{4.35}$$

It should be remarked that (4.35), which was obtained assuming $|\Gamma\Lambda/\omega_0| \ll 1$, is identical in the limit $\Gamma \to \infty$ with the equation which would have been obtained by analysing the flow within an insulating cylindrical container. The basic state is unstable if ω_1 has positive imaginary part. From (4.35) we see that instability occurs if

$$\operatorname{Re}\left\{\frac{n^2R^2(2+R\omega_0)}{2(1+R\omega_0)} + \frac{q_c\delta(R^2-1)\,(m^2+n^2)\,m^2\tau^2}{2\omega_0(1+R\omega_0)\,[R(m^2+n^2)-m]}\right\} < 0. \tag{4.36}$$

The first term in (4.36) represents the effect of internal Ohmic dissipation within the fluid while the second represents the effects of Ohmic dissipation in the fluid boundary layer near $s = 1$ and within the boundary. If the former is more important than the latter, as occurs for example if the boundary is a perfect conductor, then we recover the mode of instability discussed in § 3. With the value of the factor k, introduced in (3.23), known to be n^2R^2 for the cylindrical case, we see that the most unstable mode occurs for $n^2 = \infty$. This is very unusual: the addition of Ohmic dissipation to a neutral wave produces an instability which is strongest for short wavelengths. This is the result of the curious reversal of the role of Ohmic dissipation for these exceptional modes; instead of being most strongly stabilizing for short wavelengths, it is most strongly destabilizing. The addition of viscous dissipation to the model does not remove this curious property but changes the criterion for instability from (3.27) to

$$m = 1, \quad 1-[(1-P_m)/(1+P_m)]^2 < \tau^2(2-R)\,R, \tag{4.37}$$

where $P_m = \nu/\lambda$ is the magnetic Prandtl number. As Soward (1979) points out, the magnetic Prandtl number of the core, although much smaller than unity, is sufficiently large to preclude this mode of instability in the core. To determine the most unstable mode it is necessary to include dissipation in the basic problem (3.11)–(3.13). It should be noted however that we have found a class of wave modes which are unstable for any wavenumber n and the fact that the most unstable mode cannot be found by the present analysis does not detract from the remarkable fact that such unstable modes occur. In the following paper, Soward (1979) shows that the most unstable mode occurs for $n^2 = O(\Lambda^{-1})$, where $\Lambda \ll 1$.

Let us now consider inequality (4.36) in the case where dissipation within and near the boundary is more important than internal dissipation, as occurs, for example, if the conductivity of the container is finite. Since $|R| > 1$ and $\operatorname{Re}(q_c\delta) > 0$, (4.36) becomes simply

$$\omega_0(1+R\omega_0)\,[R(m^2+n^2)-m] < 0. \tag{4.38}$$

This inequality cannot be satisfied if $R < -1$. If $R > 1$ then $R(m^2+n^2)-m > 0$ and (4.38) may be expressed as

$$R\omega_0(1+R\omega_0) < 0$$

or, using (3.15),

$$(1+a\tau^2)^{\frac{1}{2}} \mp 1 < 0, \tag{4.39}$$

where $a = mR(mR-2)$. Instability is possible only if we choose the top sign, thus selecting the slow mode. Further, we must require

$$m = 1, \quad 1 < R < 2 \tag{4.40}$$

to obtain instability. This criterion is identical to that found previously but now it is the *slow* mode which is unstable, rather than the fast mode. As before, the unstable mode has a westward phase speed. Now the most unstable mode occurs for $n^2 = 0$ and $\omega_0 = 0$, giving a stationary wave. The boundary-layer analysis is not valid as $\omega_0 \rightarrow 0$, hence the most unstable mode is not accessible to the present analysis. The instability of this mode in a cylindrical container is due to the absence of end walls on the container. In the opposite extreme of a cylinder with end walls but no side walls, Soward (1979) finds this mode to be stable. It would be of interest to determine the stability of the slow MC modes in a finite cylinder of arbitrary aspect ratio.

4.2. *A spherical container*

If the container is spherical, the analysis becomes a direct extension of the study by Malkus (1967) and we may use his results to obtain solutions of the neutral-eigenmode problem (3.11)–(3.13). Our parameter R is related to his λ by $\lambda R = 2$. The general solution of (3.11)–(3.13) for a spherical container can be expressed as a product of associated Legendre polynomials P_n^m. We wish to determine whether any of these modes can be diffusively unstable. Relying upon the results for the cylindrical container, we shall assume that the ordinary modes, for which $R(mR-2) \geqslant 0$, are all stable and concentrate our search on those exceptional modes for which $m = 1$ and $1 < R < 2$.

Before proceeding with the stability calculation, let us determine whether such exceptional modes for the sphere do exist. The eigenvalues R are solutions of the equation

$$(n+1)RP_{n-1}^1(1/R) = (n+R)P_n^1(1/R) \tag{4.41}$$

(see equation (3.12) of Malkus 1967). Analytical and numerical solutions of (4.41) reveal that exceptional modes do exist. The eigenvalues for $3 \leqslant n \leqslant 15$ are listed in table 1.

For each eigenmode, stability is determined by (4.11), where I is given by (4.24). With $\tau \ll 1$, $\omega_0 = -\frac{1}{2}(2-R)\tau^2$. If Λ is sufficiently small that internal dissipation is negligible compared with dissipation within and near the boundary, (4.11) simplifies to

$$\omega_1 = -\Lambda^{\frac{1}{2}}R(2-R)I/4\tau. \tag{4.42}$$

The basic state is unstable if $\operatorname{Im}\omega_1 > 0$ or if $\operatorname{Im} I < 0$. Determination of the sign of $\operatorname{Im} I$ requires numerical evaluation of a complicated integral. In light of the fact that the most unstable modes in the cylinder are those for which n^2 is small and because of the burgeoning complexity of the modes as $n-m$ increases, attention was limited to the first, second and fourth modes listed in table 1. The details of these

n	R	n	R
3	1·32455532	11	1·94009902
4	1·17094380	12	1·01890886
5	1·10742106	12	1·10680611
5	1·91278275	12	1·30136699
6	1·07429604	12	1·71038608
6	1·53130127	13	1·01616112
7	1·05463996	13	1·09034087
7	1·35708758	13	1·24924304
8	1·04195488	13	1·56157666
8	1·25972834	14	1·01397425
8	1·92718424	14	1·07747791
9	1·03326655	14	1·21010226
9	1·19875507	14	1·45815096
9	1·64190223	14	1·94956780
10	1·02704389	15	1·01220478
10	1·15763392	15	1·06722278
10	1·47823819	15	1·17983972
11	1·02242893	15	1·38261537
11	1·12840874	15	1·75691196
11	1·37343860		

TABLE 1. Exceptional modes for the sphere ($m = 1,\ 1 < R < 2$).

calculations are given in appendix C. In each case, the mode was found to be *stable* for all possible values of the conductivity ratio Γ. The stability of these modes appears to be due in part to the stabilizing influence of the spherical boundaries, which, in contrast to those of the cylinder studied in §4.1, are not parallel to the axis of rotation.†

5. The influence of density gradients

In the previous sections we have found that the simple basic state of a homogeneous rigidly rotating fluid with a uniform axial electric current is prone to two distinct types of hydromagnetic instability. This result appears to conflict with that of Eltayeb & Kumar (1977), who found that a top-heavy density gradient is required to destabilize the basic state within a spherical container. To try to clarify this issue, we shall now briefly consider the effects of adding a density gradient to our model. Specifically, we shall determine how the consistency condition (4.11) is altered as the fluid becomes inhomogeneous. We shall restrict our attention to a spherical container and generalize the basic state (2.10) to include a radial gravitational force and a radial density gradient, both proportional to the radius:

$$\mathbf{g} = -g\mathbf{r}/d, \quad \nabla C^0 = \alpha\beta\mathbf{r}/d, \tag{5.1}$$

where g is the acceleration due to gravity at the surface $r = d$ of the sphere, α is the coefficient of volume expansion (assumed positive) and β is the negative of the temperature gradient at $r = d$ produced by a uniform volumetric distribution of

† An attempt to obtain the solutions for the cylinder and sphere as special cases of a spheroidal container was thwarted by the fact that an important approximation leading to (C4) is not uniformly valid as the spheroid approaches a cylindrical shape.

heat sources within the quiescent fluid. A positive value of β means the density increases with radius.

The dimensionless governing equations (3.2)–(3.4) must be modified by the addition of the term $-GC\mathbf{r}$ to the right-hand side of (3.2), where $\mathbf{r}$ is now dimensionless and

$$G = g/\Omega^2 d, \tag{5.2}$$

and by the addition of the thermal diffusion equation

$$-i\omega C + \mathbf{v}.\nabla C^0 = K\nabla^2 C, \tag{5.3}$$

where

$$K = \kappa/\Omega d^2. \tag{5.4}$$

The container is assumed to have arbitrary electrical conductivity, as before, but infinite thermal conductivity, giving

$$C = 0 \quad \text{on} \quad r = 1. \tag{5.5}$$

Assuming the effects of Ohmic diffusion to be small throughout most of the fluid, (3.3) reduces to $\mathbf{v} = -(\omega/m)\,\mathbf{B}$ and (5.3) becomes

$$i\omega C + K\nabla^2 C = -(\omega/m)\,(\alpha\beta d)\,rB_r. \tag{5.6}$$

We shall assume that the density gradient is sufficiently small that the neutral mode is given by the solution of (3.11)–(3.13). This limits our investigation to the influence of small buoyancy effects upon MC waves and precludes the study of full MAC waves. The density perturbation $C = C_1$ is found by solving

$$i\omega_0 C_1 + K\nabla^2 C_1 = -(\omega_0/m)\,(\alpha\beta d)\,rB_{0r}, \tag{5.7}$$

subject to condition (5.5). The effect of the density perturbation appears in (3.19) as the term $-imGC_1\mathbf{r}$ on the right-hand side. This contributes to the consistency condition (4.11), giving

$$2(1+R\omega_0)\,\omega_1 = -i\Lambda k(2+R\omega_0) + \Lambda^{\frac{1}{2}}\tau I + RI_A, \tag{5.8}$$

where

$$I_A = m^2 G\int_V [-|C_1|^2 + i(K/\omega_0)\,|\nabla C_1|^2]\,dV/\alpha\beta d\int_V |\mathbf{B}_0|^2\,dV. \tag{5.9}$$

The imaginary part of (5.8) yields

$$\operatorname{Im}\omega_1 = \tfrac{1}{2}\Lambda^{\frac{1}{2}}[(M+P)\,\tau R\omega_0 - \Lambda^{\frac{1}{2}}k(2+R\omega_0)]/(1+R\omega_0), \tag{5.10}$$

where

$$M = m^2 GK\int_V |\nabla C_1|^2\,dV/(\alpha\beta d)\,\omega_0^2\Lambda^{\frac{1}{2}}\tau\int_V |\mathbf{B}_0|^2\,dV \tag{5.11}$$

and

$$P = \operatorname{Im} I. \tag{5.12}$$

Note that

$$R\omega_0(1+R\omega_0)^{-1} = 1 \mp (1+a\tau^2)^{-\frac{1}{2}}, \tag{5.13}$$

where $a = mR(mR-2)$.

Instability occurs if $\operatorname{Im}\omega_1 > 0$ and the parameter M is positive if the fluid is top-heavy. Thus a top-heavy density gradient tends to destabilize the fluid provided that $1 \pm (1+a\tau^2)^{\frac{1}{2}} > 0$. This is the case for all fast modes and for all ordinary slow modes. However, the exceptional slow modes have $1-(1+a\tau^2)^{\frac{1}{2}} < 0$, so that they appear to be *destabilized* by a *bottom*-heavy density gradient. It should be noted that in this simple analysis we have not ascertained the concomitant behaviour of the factor P

in (5.10) and thus cannot make definitive statements about the stability properties of the various modes in the presence of density gradients. In the following paper Soward (1979) investigates the effects of buoyancy and dissipation in a particularly simple model and elucidates some of the complexities of the stability properties of MAC waves. He establishes that the paradoxical role of buoyancy in the exceptional modes does not persist if the bottom-heavy gradient is increased sufficiently far. At values of the stratification far larger than those for which the present analysis is valid, bottom-heavy density gradients become stabilizing once more. As Soward explains, this behaviour may be interpreted in terms of a totally new branch of the stability curve for the Eltayeb–Kumar model. As further corroboration we may refer to unpublished work of Acheson in the analogous problem of convection driven by magnetic buoyancy. He too finds parameter ranges where weak top-heavy density distributions are stabilizing. These results may seem less paradoxical when it is recalled that the energy for the instabilities is drawn from field-line curvatures. The addition of a small stratification can move the wave frequency into a domain where diffusion can better help to 'release the magnetic constraint' and so enhance the growth of the instability; this effect can be more significant than that of gravitational energy release or absorption.

6. Summary

In the previous sections we have investigated the stability characteristics of some simple steady magnetohydrodynamic flows within an axisymmetric container of arbitrary electrical conductivity. Our attention was focused upon rapidly rotating fluids and upon the geomagnetohydrodynamic basic state

$$\mathbf{B}(s,z) = (\mu\rho)^{\frac{1}{2}}\tau s\mathbf{1}_\phi, \quad \mathbf{v}(s,z) = 0, \quad \rho(s,z) = \text{constant}, \tag{6.1}$$

representing a rigidly rotating homogeneous fluid with a uniform axial electric current. We have found that this simple basic flow possesses some unexpected stability properties.

The analysis of §2 was essentially a distillation of previous efforts by Braginskiĭ (1967), Malkus (1967), Acheson (1972) and others to find criteria for instability of state (6.1) in the absence of dissipative effects. These are referred to as dynamic instabilities. In addition to providing background for the analysis of §3, the aim of this section is to demonstrate clearly the principal shortcomings of the dissipationless studies: the instability occurs only for a relatively large magnetic field, it also occurs for arbitrarily short wavelength and it results in a single overturning rather than persistent motions.

The stability of a fluid of finite electrical conductivity confined within a perfectly conducting axisymmetric container was analysed in §3. A consistency condition was obtained relating the change in frequency of a dissipationless eigenmode due to resistivity of the fluid to integrals of that eigenmode and it was found that a class of modes exists which are destabilized by the introduction of Ohmic dissipation. These unstable modes, referred to as exceptional modes, have the property that the factor $a = mR(mR-2)$ is negative; a is negative provided that $m = 1$ and $1 < R < 2$. With a perfectly conducting boundary the unstable modes are inertial modes with a westward phase speed. In contrast to the dynamic instabilities, which occur only if the magnetic field strength exceeds a critical value, the diffusive instability occurs for any non-zero value of the magnetic field provided that the Ohmic diffusivity λ is sufficiently small.

The analysis of §3 was generalized in §4 to include the effects of finite conductivity of the walls. As in §3, a consistency condition was obtained relating the frequency perturbation to integrals of the dissipationless eigenmodes, valid for an axisymmetric container. The integrals were evaluated for two container shapes of particular interest: a cylinder and a sphere. It was found that for a cylindrical container the instability of inertial modes which was found in §3 can still occur with the instability strongest for short wavelengths. It was shown that the addition of viscosity to the model does not remove the instability but does modify the criterion for instability. In addition it was found that a new class of unstable modes occurs within a cylinder. These modes have the property that $mR(mR-2) < 0$ as before but the modes are now *slow* waves rather than the fast waves. These unstable slow waves travel to the west. It was established that a number of exceptional slow modes exist within the sphere but the few modes which were investigated in detail were found to be stable.

The influence of density gradients upon the unstable modes found in §§3 and 4 was investigated in §5 for a spherical container. It was found that all fast modes and the ordinary slow modes with $mR(mR-2) > 0$ have standard stability properties in response to density gradients: top-heavy gradients are destabilizing and bottom-heavy are stabilizing. However the exceptional slow modes were found to behave in just the opposite fashion: they are stabilized by a top-heavy gradient and destabilized by a bottom-heavy gradient! This startling property may be explained in terms of a new branch on the standard stability diagram (see figure 2 of Soward 1979).

There is a sense in which there is no dynamo problem for a perfectly conducting, simply connected fluid mass: the net flux from the mass can neither strengthen nor decay, no matter what the fluid motions in its interior. One would normally expect that, as in a rigid conductor, the addition of resistivity will cause the field to decay. However, kinematic dynamos do exist and have been constructed theoretically (e.g. G. O. Roberts, reported in Roberts 1971; Perkeris, Accad & Shkoller 1973; Kumar & Roberts 1975). Their emergent flux intensifies rather than decreases when a small but positive resistivity is present. In this sense, the dynamo is a 'negative-resistivity' phenomenon, in analogy with Starr's (1968) 'negative-viscosity' phenomenon in meteorology. In the present paper we have found a surprisingly simple flow which exhibits an unusual instability which also may be characterized as a negative-resistivity effect because for certain exceptional modes resistivity intensifies the perturbation rather than damping it. One cannot help but wonder whether the negative-resistivity character of the dynamo might not be rooted in a simple instability of the sort considered in the present paper.

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Appendix A

In this appendix we shall generalize the analysis of §2 by analysing the dynamic stability of the basic state (2.1) rather than the simpler state (2.2). The stability of this state is examined by writing

$$\mathbf{B} = \mathbf{B}^0 + \mathbf{B}', \quad \mathbf{v} = \mathbf{v}^0 + \mathbf{v}', \quad C = C^0 + C'$$

and neglecting the squares and products of all perturbation (primed) fields. Using the Boussinesq approximation and neglecting all dissipation, the linearized equations are

$$\nabla . \mathbf{B}' = \nabla . \mathbf{v}' = 0, \tag{A 1}$$

$$\partial \mathbf{B}'/\partial t + \zeta \partial_1 \mathbf{B}'/\partial \phi - (\mu \rho_r)^{\frac{1}{2}} \tau \partial_1 \mathbf{v}'/\partial \phi - s[\mathbf{B}' . \nabla \zeta - (\mu \rho_r)^{\frac{1}{2}} \mathbf{v}' . \nabla \tau] \mathbf{1}_\phi = 0, \tag{A 2}$$

$$\partial C'/\partial t + \zeta \partial C'/\partial \phi + \mathbf{v}' \nabla . C^0 = 0, \tag{A 3}$$

$$\partial \mathbf{v}'/\partial t + \mathbf{v}^0 . \nabla \mathbf{v}' + \mathbf{v}' . \nabla \mathbf{v}^0 + 2\mathbf{\Omega} \times \mathbf{v}' = -\nabla p' + C' \mathbf{g} + [\mathbf{B}^0 . \nabla \mathbf{B}' + \mathbf{B}' . \nabla \mathbf{B}^0]/\mu \rho_r, \tag{A 4}$$

where $\mathbf{g}$ is the acceleration due to gravity and $\partial_1/\partial \phi$ is differentiation holding $\mathbf{1}_s$ and $\mathbf{1}_\phi$ fixed.

Following Frieman & Rotenberg (1960), Braginskiĭ (1967) solved (A 1)–(A 3) by writing the perturbations in terms of a Lagrangian displacement vector $\boldsymbol{\eta}'$:

$$\left.\begin{aligned} \mathbf{B}' &= \nabla \times (\boldsymbol{\eta}' \times \mathbf{B}^0) = \tau \partial_1 \boldsymbol{\eta}'/\partial \phi - s(\boldsymbol{\eta}' . \nabla \tau) \mathbf{1}_\phi, \\ \mathbf{v}' &= \partial \boldsymbol{\eta}'/\partial t + \nabla \times (\boldsymbol{\eta}' \times \mathbf{v}^0) = \partial \boldsymbol{\eta}'/\partial t + \zeta \partial_1 \boldsymbol{\eta}'/\partial \phi - s(\boldsymbol{\eta}' . \nabla \zeta) \mathbf{1}_\phi, \\ C' &= -\boldsymbol{\eta}' . \nabla C^0. \end{aligned}\right\} \tag{A 5}$$

The vector $\boldsymbol{\eta}'$, which is assumed to be solenoidal, must satisfy the momentum equation (A 4) in the form

$$(\partial/\partial t + \zeta \partial_1/\partial \phi)[\partial/\partial t + \zeta \partial_1/\partial \phi + 2(\Omega + \zeta) \mathbf{1}_z \times] \boldsymbol{\eta}' = -\nabla p' + \mathbf{F}(\boldsymbol{\eta}'), \tag{A 6}$$

where

$$\mathbf{F}(\boldsymbol{\eta}') = \tau^2 (\partial_1/\partial \phi + 2\mathbf{1}_z \times) \partial_1 \boldsymbol{\eta}'/\partial \phi - \boldsymbol{\alpha}(\boldsymbol{\eta}'), \tag{A 7}$$

$$\begin{aligned} \boldsymbol{\alpha}(\boldsymbol{\eta}') &= (\boldsymbol{\eta}' . \nabla C^0) \mathbf{g} + 2\Omega (s \boldsymbol{\eta}' . \nabla \zeta_*) \mathbf{1}_s \\ &= (\mathbf{g} . \boldsymbol{\eta}') \nabla C^0 + 2\Omega (s \boldsymbol{\eta}' . \mathbf{1}_s) \nabla \zeta_* \end{aligned} \tag{A 8}$$

and

$$\zeta_* = \zeta + (\zeta^2 - \tau^2)/2\Omega. \tag{A 9}$$

The two statements (A 8) defining the tensor $\boldsymbol{\alpha}$ are equivalent because the basic state (2.1) is in magnetostatic balance, requiring

$$2\Omega s \partial \zeta_*/\partial z = \mathbf{1}_\phi . \mathbf{g} \times \nabla C^0, \tag{A 10}$$

a generalization of the thermal-wind equation.

Again seeking the mode of maximum instability by introducing a right-handed system of local Cartesian co-ordinates (ν, ξ, ϕ), where $\mathbf{1}_\nu$ makes an arbitrary angle χ with $\mathbf{1}_z$ as shown in figure 1, and assuming the perturbations to be proportional to $\exp[i(m\phi - \omega t)]$, we may write (A 6) in component form as

$$[m^2\tau^2 - (\omega - m\zeta)^2 + \alpha_{\nu\nu}] \eta'_\nu + \alpha_{\nu\xi} \eta'_\xi + 2i[m^2\tau^2 + (\Omega + \zeta)(\omega - m\zeta)] \eta'_\phi \sin \chi = -\partial p'/\partial \nu, \tag{A 11}$$

$$\alpha_{\xi\nu}\eta'_\nu + [m^2\tau^2 - (\omega - m\zeta)^2 + \alpha_{\xi\xi}]\,\eta'_\xi + 2i[m^2\tau^2 + (\Omega+\zeta)(\omega - m\zeta)]\,\eta'_\phi \cos\chi = -\partial p'/\partial\xi, \quad \text{(A 12)}$$

$$-2i[m\tau^2 + (\Omega+\zeta)(\omega - m\zeta)]\,[\eta'_\nu \sin\chi + \eta'_\xi \cos\chi] + [m^2\tau^2 - (\omega - m\zeta)^2]\,\eta'_\phi = -imp'/s. \quad \text{(A 13)}$$

The condition that $\boldsymbol{\eta}'$ be solenoidal is

$$\partial(s\eta'_\nu)/\partial\nu + \partial(s\eta'_\xi)/\partial\xi + im\eta'_\phi = 0. \quad \text{(A 14)}$$

We now seek a condition for the system to be stable at a local point P to disturbances whose wavelength is arbitrarily short in the ξ direction. That is, we let $\partial/\partial\xi = l$ and consider $l \to \infty$. From (A 14), $\eta'_\xi = O(l^{-1}\eta_\phi)$ and, from (A 12), $p' = O(l^{-1}\eta_\phi)$. Thus to leading order in l^{-1}, (A 11) and (A 13) reduce to

$$[m^2\tau^2 - (\omega - m\zeta)^2 + \alpha_{\nu\nu}]\,\eta'_\nu + 2i[m\tau^2 + (\Omega+\zeta)(\omega - m\zeta)]\,\eta'_\phi \sin\chi = 0, \quad \text{(A 15)}$$

$$-2i[m\tau^2 + (\Omega+\zeta)(\omega - m\zeta)]\,\eta'_\nu \sin\chi + [m^2\tau^2 - (\omega - m\zeta)^2]\,\eta'_\phi = 0. \quad \text{(A 16)}$$

The condition that modes of this form exist is thus purely algebraic (cf. Gilman 1970):

$$[m^2\tau^2 - (\omega - m\zeta)^2]\,[m^2\tau^2 - (\omega - m\zeta)^2 + \alpha_{\nu\nu}] = 4[m\tau^2 + (\Omega+\zeta)(\omega - m\zeta)]^2 \sin^2\chi. \quad \text{(A 17)}$$

Neglecting inertial terms reduces this to

$$m^2\tau^2(m^2\tau^2 + \alpha_{\nu\nu}) = 4[m\tau^2 + \Omega(\omega - m\zeta)]^2 \sin^2\chi. \quad \text{(A 18)}$$

A sufficient condition for instability is

$$m^2\tau^2 < -\alpha_{\nu\nu} \quad \text{(A 19)}$$

for some direction $\mathbf{1}_\nu$ at some point P. According to (A 8),

$$\alpha_{\nu\nu} = g_\nu \partial C^0/\partial\nu + 2\Omega s(\partial\zeta_*/\partial\nu)\sin\chi, \quad \text{(A 20)}$$

where $g_\nu = \mathbf{g}\,.\,\mathbf{1}_\nu$. It is apparent that the most unstable mode has a zonal wavenumber $m = 1$, giving

$$\tau^2 + g_\nu \partial C^0/\partial\nu + 2\Omega s(\partial\zeta_*/\partial\nu)\sin\chi < 0. \quad \text{(A 21)}$$

If the fluid is homogeneous, then $C^0 = 0$ and, from (A 10), $\zeta_* = \zeta_*(s)$ and (A 21) reduces to

$$\tau^2 + 2\Omega s(d\zeta_*/ds)\sin^2\chi < 0. \quad \text{(A 22)}$$

In this case the most unstable direction is $\mathbf{1}_\nu = \mathbf{1}_s$, giving for $\Omega > 0$

$$2\Omega\, d\zeta/dz < s\, d(\tau^2/s)/ds. \quad \text{(A 23)}$$

In the absence of shear, (A 23) reduces to (2.9).

It should be emphasized that the Cartesian analysis of this appendix does not model the curvature terms accurately. Hence (A 21) is not valid near the polar axis (s small). That is, while (A 21) predicts instability near the polar axis for any top-heavy density distribution, a stability analysis including curvature effects might not.

Appendix B

In this appendix we shall generalize the analysis of §3 leading to the consistency condition (3.24) for the basic state (2.1). Assuming harmonic dependence on ϕ and t, the dimensionless problem is

$$-i\omega\mathbf{v}' + \mathbf{v}^0.\nabla\mathbf{v}' + \mathbf{v}'.\nabla\mathbf{v}^0 + 2\mathbf{1}_z \times \mathbf{v}' = -\nabla p' + C'\mathbf{g} + \mathbf{B}^0.\nabla\mathbf{B}' + \mathbf{B}'.\nabla\mathbf{B}^0, \quad \text{(B 1)}$$

$$-i\omega\mathbf{B}' = \nabla \times [\mathbf{v}' \times \mathbf{B}^0 + \mathbf{v}^0 \times \mathbf{B}' - \Lambda\nabla \times \mathbf{B}'], \quad \text{(B 2)}$$

$$i(\omega - m\zeta)\, C' = \mathbf{v}'.\nabla C^0, \quad \text{(B 3)}$$

$$\nabla.\mathbf{v}' = \nabla.\mathbf{B}' = 0, \quad \text{(B 4)}$$

with

$$\mathbf{n}.\mathbf{B}' = 0, \quad \mathbf{n} \times (\nabla \times \mathbf{B}') = 0, \quad \mathbf{n}.\mathbf{v}' = 0 \quad \text{on} \quad W, \quad \text{(B 5)}$$

where we have used the scaling of §3 plus $\Omega^2 d$ for gravity; C^0 and C' were dimensionless to start with.

Following Braginskiĭ & Roberts (1975), the formalism of appendix A may be generalized by introducing a separate displacement vector $\boldsymbol{\eta}_B'$ for magnetic field lines and replacing (A 5)$_1$ by

$$\mathbf{B}' = \nabla \times (\boldsymbol{\eta}_B' \times \mathbf{B}^0), \quad \nabla.\boldsymbol{\eta}_B' = 0 \quad \text{(B 6)}$$

though preserving (A 5)$_2$ and (A 5)$_3$. These assumed forms satisfy (B 3) and give (B 4) automatically. The remaining equations give

$$-(\omega - m\zeta)[\omega - m\zeta + 2i(1+\zeta)\,\mathbf{1}_z \times\,]\,\boldsymbol{\eta}'$$
$$= -\nabla p' - s(\boldsymbol{\eta}' - \boldsymbol{\eta}_B')\,\nabla(\tau^2) - \tau^2 m(m - 2i\,\mathbf{1}_z \times\,)\,\boldsymbol{\eta}_B' - \boldsymbol{\alpha}(\boldsymbol{\eta}'), \quad \text{(B 7)}$$

$$-i\omega\boldsymbol{\eta}_B' \times \mathbf{B}^0 = \mathbf{v}' \times \mathbf{B}^0 + \mathbf{v}^0 \times \mathbf{B}' - \Lambda(\nabla \times \mathbf{B}' - \nabla\Phi'), \quad \text{(B 8)}$$

where $\boldsymbol{\alpha}$ is given by (A 8) and Φ' is a function of integration. To determine Φ' we take the $\mathbf{1}_\phi$ component of (B 8):

$$im\Phi' = s\mathbf{1}_\phi.\nabla \times \mathbf{B}'. \quad \text{(B 9)}$$

Using (B 6) once more, we may write (B 9) as

$$\Phi' = s[\partial(\tau\eta_{Bs}')/\partial z - \partial(\tau\eta_{Bz}')/\partial s]. \quad \text{(B 10)}$$

Now the cross-product of (B 8) with $\mathbf{1}_\phi$ may be expressed as

$$is\tau(\omega - m\zeta)\,(\boldsymbol{\eta}' - \boldsymbol{\eta}_B')_m = -\Lambda\mathbf{1}_\phi \times (\nabla \times \mathbf{B}' - \nabla\Phi'), \quad \text{(B 11)}$$

where the subscript m denotes the meridional part. Since both $\boldsymbol{\eta}'$ and $\boldsymbol{\eta}_B'$ are solenoidal, we may write

$$\boldsymbol{\eta}' = \boldsymbol{\eta}_B' + \nabla \times \mathbf{G}'. \quad \text{(B 12)}$$

The vector $\mathbf{A}' = \nabla \times \mathbf{B}' - \nabla\Phi'$ has no azimuthal part, allowing us to write

$$im\mathbf{1}_\phi \times \mathbf{A}' = s(\nabla \times \mathbf{A}')_m.$$

Since the cross-product of two meridional vectors has no meridional component, we have

$$(\nabla \times \mathbf{G}')_m = \Lambda\left[\nabla \times \left(\frac{\nabla \times \mathbf{B}' - \nabla\Phi'}{m\tau(\omega - m\zeta)}\right)\right]_m, \quad \text{(B 13)}$$

or

$$\mathbf{G}' = \Lambda \frac{\nabla \times \mathbf{B}' - \nabla \Phi'}{m\tau(\omega - m\zeta)} + \nabla \Psi', \tag{B 14}$$

where Ψ' is an arbitrary function. The following analysis is simplified if we choose

$$\Psi' = \Lambda \Phi' / m\tau(\omega - m\zeta), \tag{B 15}$$

which gives

$$\mathbf{G}' = \frac{\Lambda}{m}\left[\frac{\nabla \times \mathbf{B}'}{\tau(\omega - m\zeta)} - \frac{is}{m}(\mathbf{1}_\phi . \nabla \times \mathbf{B}')\nabla\left(\frac{1}{\tau(\omega - m\zeta)}\right)\right]. \tag{B 16}$$

Now (B 7) may be written as

$$\begin{aligned}[(\omega - m\zeta)^2 - m^2\tau^2]\,\boldsymbol{\eta}' + 2i[(\omega - m\zeta)(1+\zeta) + m\tau^2]\,\mathbf{1}_z \times \boldsymbol{\eta}' \\ - 2(\boldsymbol{\eta}' . \nabla\zeta_*)\, s\mathbf{1}_s - (\boldsymbol{\eta}' . \nabla C^0)\,\mathbf{g} - \nabla p' \\ = \tau^2[-m^2 + 2im\mathbf{1}_z \times\,](\boldsymbol{\eta}' - \boldsymbol{\eta}_B') + [(\boldsymbol{\eta}' - \boldsymbol{\eta}_B') . \nabla(\tau^2)]\, s\mathbf{1}_s.\end{aligned} \tag{B 17}$$

The boundary conditions may be expressed as

$$\mathbf{n} . \boldsymbol{\eta}' = 0, \quad \mathbf{n} . \boldsymbol{\eta}_B' = 0, \quad \mathbf{n} \times \nabla \times \mathbf{B}' = 0 \quad \text{on} \quad W. \tag{B 18}$$

Note that from $(\text{B } 18)_3$

$$\mathbf{n} \times \mathbf{G}' = 0 \quad \text{on} \quad W. \tag{B 19}$$

The first step is to obtain neutral dissipationless modes by neglecting Λ and solving the eigenvalue problem

$$\left.\begin{aligned}[(\omega_0 - m\zeta)^2 - m^2\tau^2]\,\boldsymbol{\eta}_0 + 2i[(\omega_0 - m\zeta)(1+\zeta) + m\tau^2]\,\mathbf{1}_z \times \boldsymbol{\eta}_0 \\ - 2(\boldsymbol{\eta}_0 . \nabla\zeta_*)\, s\mathbf{1}_s - (\boldsymbol{\eta}_0 . \nabla C^0)\,\mathbf{g} - \nabla p_0 = 0, \\ \nabla . \boldsymbol{\eta}_0 = 0 \\ \text{with} \qquad \mathbf{n} . \boldsymbol{\eta}_0 = 0 \quad \text{on} \quad W,\end{aligned}\right\} \tag{B 20}$$

where ω_0 is the eigenfrequency corresponding to the eigenvector $\boldsymbol{\eta}_0$. We shall assume that the modes obtained from (B 20) do not contain resonant surfaces on which $\zeta = \omega_0/m$.† These surfaces are avoided in the special case $\zeta = 0$ by excluding any stationary mode $\omega_0 = 0$. We shall further assume that the eigensolutions are not localized as in appendix A but extend throughout the fluid.

With $\Lambda = 0$, the displacements $\boldsymbol{\eta}'$ and $\boldsymbol{\eta}_B'$ are identical as in appendix A. When Λ is allowed to increase from zero, $\boldsymbol{\eta}'$ and $\boldsymbol{\eta}_B'$ each change and by different amounts. Also the basic frequency ω_0 is perturbed. Thus we may write

$$\left.\begin{aligned}\boldsymbol{\eta}' = \boldsymbol{\eta}_0 + \boldsymbol{\eta}_1, \quad \boldsymbol{\eta}_B' = \boldsymbol{\eta}_0 + \boldsymbol{\eta}_1 - \nabla \times \mathbf{G}', \\ p' = p_0 + p_1, \quad \omega = \omega_0 + \omega_1.\end{aligned}\right\} \tag{B 21}$$

It is tempting to replace $\mathbf{G}'$ in (B 21) by $\mathbf{G}_0$, where $\mathbf{G}_0$ is given by (B 16) and (B 6) with $\omega = \omega_0$ in (B 16) and $\boldsymbol{\eta}_B' = \boldsymbol{\eta}_0$ in (B 6). Unfortunately this substitution is not uniformly valid and its use at this point in the analysis could lead to erroneous results.

† This assumption precludes the resistive tearing mode of plasma physics (Furth *et al.* 1963); the relevance of this mode to the geomagnetic secular variation is currently under investigation. If present, the growth rate of the tearing mode should be asymptotically larger than the instability considered here and in §3, but also should be asymptotically smaller than the instability considered in §4.

That is, while $\boldsymbol{\eta}_1$ and $\nabla \times \mathbf{G}'$ are each of order Λ throughout most of the fluid, each becomes larger that $O(\Lambda)$ in thin magnetic boundary layers near W. We shall indeed replace $\mathbf{G}'$ by $\mathbf{G}_0$, but at the appropriate point in the analysis. First substitute (B 21) into (B 17) and, making use of (B 20), obtain

$$\left.\begin{aligned}
&[(\omega_0-m\zeta)^2-m^2\tau^2]\boldsymbol{\eta}_1+2i[(\omega_0-m\zeta)(1+\zeta)+m\tau^2]\mathbf{1}_z\times\boldsymbol{\eta}_1\\
&\quad-2(\boldsymbol{\eta}_1.\nabla\zeta_*)s\mathbf{1}_s-(\boldsymbol{\eta}_1.\nabla C^0)\mathbf{g}-\nabla p_1\\
&\quad=-2\omega_1(\omega_0-m\zeta)\boldsymbol{\eta}_0-2i\omega_1(1+\zeta)\mathbf{1}_z\times\boldsymbol{\eta}_0\\
&\quad+\tau^2[-m^2+2im\mathbf{1}_z\times]\nabla\times\mathbf{G}'+[(\nabla\times\mathbf{G}').\nabla(\tau^2)]s\mathbf{1}_s,\\
&\qquad\nabla.\boldsymbol{\eta}_1=0\\
\text{with}\quad&\qquad\mathbf{n}.\boldsymbol{\eta}_1=0\quad\text{on}\quad W.
\end{aligned}\right\}\tag{B 22}$$

The consistency condition is obtained by scalar multiplying (B 22)$_1$ by $\boldsymbol{\eta}_0^*$ and integrating the result over the volume V of the fluid. Following appendix A, the consistency integral may be expressed as

$$\begin{aligned}
2\omega_1\int_V&[(\omega_0-m\zeta)|\boldsymbol{\eta}_0|^2+i(1+\zeta)\mathbf{1}_z.\boldsymbol{\eta}_0\times\boldsymbol{\eta}_0^*]\,dV\\
&=\int_V\{\nabla\times\mathbf{G}'.[s\eta_{0s}^*\nabla(\tau^2)-(\omega_0-m\zeta)^2\boldsymbol{\eta}_0^*+2i(\omega_0-m\zeta)(1+\zeta)\mathbf{1}_z\times\boldsymbol{\eta}_0^*]\}\,dV.
\end{aligned}\tag{B 23}$$

Using the divergence theorem and (B 19), we obtain an alternative and more useful form:

$$\begin{aligned}
2\omega_1\int_V&[(\omega_0-m\zeta)|\boldsymbol{\eta}_0|^2+i(1+\zeta)\mathbf{1}_z.\boldsymbol{\eta}_0\times\boldsymbol{\eta}_0^*]\,dV\\
&=\int_V\{\mathbf{G}'.\nabla\times[s\eta_{0s}^*\nabla(\tau^2)-(\omega_0-m\zeta)^2\boldsymbol{\eta}_0^*\\
&\qquad+2i(\omega_0-m\zeta)(1+\zeta)\mathbf{1}_z\times\boldsymbol{\eta}_0^*]\}\,dV.
\end{aligned}\tag{B 24}$$

The advantage of (B 24), compared with (B 23), is that $\mathbf{G}'$ is not differentiated. We wish to determine ω_1 correctly to order Λ. Writing (B 6) as

$$\mathbf{B}'=im\tau\boldsymbol{\eta}_B'-(\boldsymbol{\eta}_B'.\nabla\tau)s\mathbf{1}_\phi,$$

it is apparent that $\mathbf{B}'=O(1)$ everywhere within the fluid. With perfectly conducting boundaries, the boundary layers in the fluid near W are weaker than for finitely conducting boundaries, specifically

$$\boldsymbol{\eta}_B'=\boldsymbol{\eta}_0+O(\Lambda^{\frac{1}{2}}).$$

Thus we have

$$\mathbf{B}'=im\tau\boldsymbol{\eta}_0-(\boldsymbol{\eta}_0.\nabla\tau)s\mathbf{1}_\phi+O(\Lambda^{\frac{1}{2}}).\tag{B 25}$$

Now $\mathbf{G}'$ involves derivatives of $\mathbf{B}'$, which magnify the error in (B 25) within thin boundary layers. However, we need only the integral of $\mathbf{G}'$ in (B 24) and the integral of the $O(1)$ error over a thin boundary layer produces at most an $O(\Lambda^{\frac{1}{2}})$ error in the evaluation of that integral. [The differentiation in (B 16) and the integration in (B 24) effectively cancel each other for purposes of error estimation.] We may now replace $\mathbf{G}'$ by

$$\mathbf{G}_0=\frac{-\Lambda}{m\tau(\omega_0-m\zeta)}\left\{\nabla^2(\boldsymbol{\eta}_0\times\mathbf{B}^0)+\frac{is[\mathbf{1}_\phi.\nabla^2(\boldsymbol{\eta}_0\times\mathbf{B}^0)]\nabla[\tau(\omega_0-m\zeta)]}{m\tau(\omega_0-m\zeta)}\right\}\tag{B 26}$$

in (B 24), making a negligible error in doing so. Now (B 24) and (B 26) provide formulae for calculating the perturbation frequency ω_1 in terms of the neutral eigenmode (η_0, p_0, ω_0). Had we assumed the container walls to have non-zero resistivity, it would not have been possible to obtain such formulae without detailed analysis of the boundary layers near W. Such analysis is carried out in §4.

Appendix C

The purpose of this appendix is to simplify the integral I, given by (4.24), for a spherical container and to evaluate its imaginary part for several eigenvalues. With spherical co-ordinates

$$\beta = r, \quad \beta_0 = 1, \quad h_\beta = 1, \quad \mathbf{1}_\beta = \mathbf{1}_r, \quad \gamma = 0,$$
$$h_\gamma = r, \quad \psi = \cos\theta, \quad s = r\sin\theta, \quad z = r\cos\theta,$$
$$dW = 2\pi\sin\theta\, d\theta = -2\pi\, dz, \quad dV = 2\pi r^2 \sin\theta\, dr\, d\theta = \pi\, d(s^2)\, dz.$$

The integral I may be expressed as

$$I = (I_4 + I_5)/I_B, \tag{C 1}$$

where

$$I_B = \int_0^1 \int_0^{1-z^2} |\mathbf{B}_0|^2\, d(s^2)\, dz, \tag{C 2}$$

$$I_4 = -\frac{\Lambda^{\frac{1}{2}}\tau}{\omega_0^2}\int_{-1}^{1}\left[q_+\frac{\partial}{\partial z}(b_+(1-z^2)^{\frac{1}{2}}) + q_-\frac{\partial}{\partial z}(b_-(1-z^2)^{\frac{1}{2}}) + \tfrac{1}{2}q_c\frac{\delta_+A_+ - \delta_-A_-}{(1-z^2)^{\frac{1}{2}}}\right]p_0^*\, dz, \tag{C 3}$$

$$I_5 = \frac{iq_c\Lambda^{\frac{1}{2}}\tau}{\omega_0}\int_{-1}^{1}[\delta_+A_+A_+^* - \delta_-A_-A_-^*]\, z\, dz. \tag{C 4}$$

In writing (C 4) we have neglected $\pm\frac{1}{2}$ compared with z/ω_0 since $1/\omega_0 = O(\tau^{-2})$ for $\tau \ll 1$.

Now

$$\omega_0 = -\tfrac{1}{2}(2-R)\tau^2, \quad q_c = \tfrac{1}{2}(1+i)(2-R)^{\frac{1}{2}}\tau\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}},$$
$$q_\pm = \tfrac{1}{2}(1+\sigma_\pm i)\tau\Lambda^{-\frac{1}{2}}|(1\pm Rz)/z|^{\frac{1}{2}}, \quad \sigma_\pm = \operatorname{sgn}[\mp z(1\pm Rz)],$$
$$\partial q_\pm/\partial z = -q_\pm[2z(1\pm Rz)]^{-1},$$
$$q_\pm\frac{\partial}{\partial z}[b_\pm(1-z^2)^{\frac{1}{2}}] \pm \tfrac{1}{2}\frac{q_c q_\pm A_\pm}{(1-z^2)^{\frac{1}{2}}} = -\tfrac{1}{2}q_c\delta_\pm(1-z^2)^{\frac{1}{2}}\left[\frac{\partial A_\pm}{\partial z} \mp \frac{A_\pm}{1\pm z} + \frac{\delta_\pm A_\pm}{2\Gamma z(1\pm Rz)}\right],$$
$$\delta_\pm = \frac{\Gamma|1\pm Rz|^{\frac{1}{2}}}{|1\pm Rz|^{\frac{1}{2}} + \frac{1}{2}[1+\sigma_\pm + i(1-\sigma_\pm)](2-R)^{\frac{1}{2}}\Gamma^{\frac{1}{2}}|z|^{\frac{1}{2}}}$$

and

$$I_4 = \frac{1+i}{(2-R)^{\frac{3}{2}}\tau^2\Gamma^{\frac{1}{2}}}\int_{-1}^{1}\left[\delta_+\left(\frac{\partial A_+}{\partial z} - \frac{A_+}{1-z}\right) + \delta_-\left(\frac{\partial A_-}{\partial z} + \frac{A_-}{1+z}\right) + \frac{\delta_+^2 A_+}{2\Gamma z(1+Rz)} + \frac{\delta_-^2 A_-}{2\Gamma z(1-Rz)}\right](1-z^2)^{\frac{1}{2}}p_0^*\, dz, \tag{C 5}$$

$$I_5 = \frac{1-i}{(2-R)^{\frac{1}{2}}\Gamma^{\frac{1}{2}}}\int_{-1}^{1}[\delta_+A_+A_+^* - \delta_-A_-A_-^*]\, z\, dz. \tag{C 6}$$

If $n-m$ is even (odd), then p_0, B_{0r} and $B_{0\theta}$ are even (odd) while $B_{0\phi}$ is odd (even). Thus as $z \to -z$, $A_\pm \to A_\mp$ if $n-m$ is even and $A_\pm \to -A_\mp$ if $n-m$ is odd. Also $\delta_\pm \to \delta_\mp$ as $z \to -z$. This is sufficient to determine that the integrands of I_4 and I_5 are even functions of z and the integrals may be collapsed to the domain [0, 1].

Since I_B is real and positive, the sign of $\operatorname{Im} I$ is the same as that of $\operatorname{Im}(I_4+I_5)$. From (3.11) and (3.14) it may be seen that it is possible to choose the phase of the solution such that B_{0r} and $B_{0\theta}$ are real while Q_0, p_0 and $B_{0\phi}$ are imaginary. Thus $A_\pm$ are purely real. Setting

$$p_0 = ir^2(R-1)(1-z^2)^{\frac{1}{2}} N(z),$$

where N is real, we may write

$$\operatorname{Im} I_4 = -\frac{2(R-1)}{(2-R)^{\frac{3}{2}}\Gamma^{\frac{1}{2}}}\int_0^1 \left[\Delta_+\left(\frac{\partial A_+}{\partial z}-\frac{A_+}{1-z}\right)+\Delta_-\left(\frac{\partial A_-}{\partial z}+\frac{A_-}{1+z}\right)+\frac{Y_+A_+ + Y_-A_-}{2z}\right] \times (1-z^2)N(z)\,dz, \quad \text{(C 7)}$$

$$\operatorname{Im} I_5 = -\frac{2}{(2-R)^{\frac{1}{2}}\Gamma^{\frac{1}{2}}}\int_0^1 [\Delta_+ A_+^2 - \Delta_- A_-^2]\,z\,dz, \quad \text{(C 8)}$$

where

$$\Delta_\pm = -\operatorname{Im}[(1-i)\,\delta_\pm] = \operatorname{Re}\delta_\pm - \operatorname{Im}\delta_\pm, \quad \text{(C 9)}$$

$$Y_\pm = \frac{\operatorname{Re}\delta_\pm^2 - \operatorname{Im}\delta_\pm^2}{\Gamma(1\pm Rz)} = \frac{2\Delta_\pm - \Gamma}{1\pm[R+\Gamma(2-R)]z}. \quad \text{(C 10)}$$

Specifically,

$$\Delta_+ = \Gamma\,\frac{1+Rz+[\Gamma(2-R)\,z(Rz-1)]^{\frac{1}{2}}}{1+[R+\Gamma(2-R)]z} \quad \text{(C 11)}$$

and

$$\Delta_- = \begin{cases} \left[\dfrac{1}{\Gamma}+\left(\dfrac{(2-R)\,z}{\Gamma(1-Rz)}\right)^{\frac{1}{2}}\right]^{-1} & \text{for } 0<z<R^{-1}, \\ \Gamma\,\dfrac{1-Rz-\{\Gamma(2-R)\,z(Rz-1)\}^{\frac{1}{2}}}{1-[R+\Gamma(2-R)]z} & \text{for } R^{-1}<z<1. \end{cases}$$

The integrals in (C 7) and (C 8) were evaluated for three eigenmodes:

(i) $m = 1, \quad n = 3, \quad R = \surd 40 - 5 \doteq 1{\cdot}32455532,$

$$N(z) = \tfrac{1}{4}(R^2-5)(5z^2-1), \quad A_\pm(z) = \tfrac{2}{9}R(2-R)(1\mp z)(15z\mp R);$$

(ii) $m = 1, \quad n = 4, \quad R \doteq 1{\cdot}17094380,$

$$N(z) = z(7-3R^2)(3-7R^2),$$
$$A_\pm(z) = (1\mp z)[7(9R^2-6R-7)z^2 \pm 42(R-1)^2 z - 3(3R^2+14R-21)];$$

(iii) $m = 1, \quad n = 5, \quad R \doteq 1{\cdot}91278275,$

$$N(z) = (R^4-14R^2+21)(21z^4-14z^2+1),$$
$$A_\pm(z) = 56(1\mp z)[3(-18R^3+21R^2+56R-63)z^3 \mp (45R^3-54R^2-147R+168)z^2 + (20R^3-25R^2-54R+63)z \pm (15R^3-20R^2-45R+54)].$$

The integrals were evaluated numerically for a wide range of values of Γ, the conductivity ratio, and each of the modes was found to be stable. As an independent check, the integrals in (C 7) and (C 8) were evaluated analytically in the limit $\Gamma \to 0$ and were found to give stable modes.

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